

## WEAK CONVERGENCE OF THE AREA OF NONPARAMETRIC $L_1$ SURFACES

BY

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**ABSTRACT.** The main purpose of this work is to obtain an analogue to a theorem of L. C. Young on the behavior of the nonparametric surface area of continuous functions. The analogue is for  $L^1$  functions of generalized bounded variation. By considering arbitrary Borel vector measures and kernels other than the area kernel, results concerning the weak behavior of measures induced by a class of sublinear functionals are obtained.

**0. Introduction.** The classes  $BVC$  and  $ACC$  of continuous functions over the open unit cube  $Q$  in  $R^m$  have been extended to the classes  $BV$  and  $AC$  of  $L^1$  functions by use of the existence of certain types of distribution derivatives. Namely, if the distribution derivative is given by a finite Borel vector measure, the function is of bounded variation. If in fact the derivative is given by a function (i.e., the measure is absolutely continuous with respect to Lebesgue measure) the function is absolutely continuous. A natural question then becomes: Which properties do these wider classes share with the continuous classes?

Here we consider a theorem concerning the behavior of the continuous, nonparametric surface area of surfaces in  $BVC$  given by L. C. Young, [6], in 1944; and develop an analogue behavior in the class  $BV$  for the generalized surface area as in [6].

In the classic area formula for  $ACC(R^2)$  functions, if the partial derivatives are replaced with difference quotients, a formula results which can be applied to any function in  $BVC$ :

$$A\alpha\beta = \iint \left[ 1 + \left[ \frac{f(x + \alpha, y) - f(x, y)}{\alpha} \right]^2 + \left[ \frac{f(x, y + \beta) - f(x, y)}{\beta} \right]^2 \right]^{1/2} dy dx.$$

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The question then becomes: How does  $A\alpha\beta$  approximate the area? Is the area  $\lim_{\alpha, \beta \rightarrow 0} A\alpha\beta$ ?

L. C. Young shows the answer is no, and gives necessary and sufficient conditions for the answer to be yes. (Compare with Saks [1, p. 182].) His condition is that there must be a Borel partition of  $Q$  into two sets  $A_1$  and  $A_2$  so that off  $A_i$  the function is absolutely continuous in the variable  $x_i$  for almost all fixed values of the other variables.

For the generalized class  $AC$  there is a similar area formula

$$\int_Q \left( 1 + \frac{d\alpha_1^2}{dL} + \frac{d\alpha_2^2}{dL} \right)^{1/2} dL$$

where  $(\alpha_1, \alpha_2)$  is the distribution derivative measure of  $f$  and  $d\alpha_i/dL$  is the general derivative [1, p. 106] of  $\alpha_i$  with respect to Lebesgue measure  $L$ .

Following Young, we want to replace the derivatives with "difference quotients" for these derivatives and find necessary and sufficient conditions for the limit to be the area when applied to a function known only to be in the class  $BV$ .

A precise statement of the result is Theorem 6 in §5 of the paper and requires a lot of development of notation. However the necessary and sufficient condition simply stated is a direct analogue of Young's condition.

The area will be given as the limit if and only if the cube  $Q$  can be partitioned into sets  $A_1, \dots, A_m$  so that off  $A_i$  the  $i$ th component of the distribution derivative measure is absolutely continuous.

By considering arbitrary Borel vector measures and kernels other than the area kernel, the results of §1 through 5 are results concerning the weak behavior of measures induced from vector measures by a class of sublinear functionals.

**1. The integral average of a measure.** In this section we make precise the idea of "difference quotient" for a measure by introducing an averaging process for measures which is an analogue to taking the integral average of a function.

For notational convenience we introduce the following definitions and conventions.

- (1)  $L$  will always denote Lebesgue measure.
- (2)  $M$  will always denote the collection of all finite Borel measures on the open unit cube  $Q$  in  $R^m$ .
- (3) For each  $h > 0$  let  $Q_h = \times_{i=1}^m [0, h]$  be the closed  $h$  cube in  $R^m$ , and let  $K_h(x) = \chi_{Q_h}(x) + h^m$  (where  $\chi$  indicates the characteristic function).
- (4) By  $\kappa[ \ , \ ]$  we denote a mapping from  $M \times R^+$  into the continuous functions from  $R^m$  into  $R$  given by

$$\kappa[\sigma, h](x) = \int_{R^m} K_h(y - x) d\sigma(y).$$

Note that  $\kappa[\sigma, h](x) = \sigma I_h(x) / L I_h(x)$  where  $I_h(x)$  is a cube of side length  $h$  containing  $x$ . Thus it is  $\kappa[\sigma, h](x)$  which corresponds to the difference quotients for functions. Further, as  $h$  tends to zero, by Lebesgue's theorem [1, p. 115], we have that for almost all  $x$ ,  $\lim_{h \rightarrow 0} \kappa[\sigma, h](x) = d\sigma/dL$ .

(5)  $\mu[ , ]$  will denote a mapping from  $M \times R^+$  into  $M$  given by

$$\mu[\sigma, h](E) = \int_E \kappa[\sigma, h](x) dL(x)$$

for each Borel set  $E$ .

Similar constructions can be found in [4, p. 167], with continuous kernels  $K_h$ .

If  $\sigma$  is supported inside  $Q$  then  $\mu[\sigma, h]$  is supported inside an  $h$ -neighborhood of  $Q$ . Further, for  $h > 0$ ,  $\mu[\sigma, h](R^m) = \sigma(R^m)$ ; since

$$\begin{aligned} \mu[\sigma, h](R^m) &= \int_{R^m} \int_{R^m} K_h(y - x) d\sigma(y) dL(x) \\ &= \int_{R^m} \int_{R^m} K_h(x - y) dL(x) d\sigma(y) \\ &= \int_{R^m} h^m h^{-m} d\sigma(y) = \sigma(R^m). \end{aligned}$$

As  $h$  tends to zero, supports for  $\mu[\sigma, h]$  tend to  $Q$  and we have

$$\lim_{h \rightarrow 0} \mu[\sigma, h](Q) = \sigma(Q).$$

A sequence of measures  $\sigma_k$  converges weakly to a measure  $\sigma$  if for each real valued continuous function  $f$  of compact support we have  $\int f d\sigma_k \rightarrow \int f d\sigma$ . For probability measures this is equivalent to  $\liminf_k \sigma_k(G) \geq \sigma(G)$  for each open set  $G$  [2, p. 11]. This equivalence will also hold if  $\sigma_k$  are positive and  $\lim_k \sigma_k(Q) = \sigma(Q)$ .

**THEOREM 1.** *As  $h$  tends to zero in  $R^+$ ,  $\mu[\sigma, h]$  converges weakly to  $\sigma$ .*

**PROOF.** See [4] and replace "uniform convergence" with "dominated convergence" in the proof of Theorem 4.

This averaging process is extended to  $n$ -dimensional vector measures in  $\times_{i=1}^n M$  as follows:  $\mu[ , ]: \times_{i=1}^n M \times \times_{i=1}^n R_+ \rightarrow \times_{i=1}^n M$  is defined for each  $\bar{\sigma} = (\sigma^1, \dots, \sigma^n)$  in  $\times_{i=1}^n M$  and vector  $\bar{h} = (h^1, \dots, h^n)$  in  $R_+^n$  by  $\mu[\bar{\sigma}, \bar{h}] = \sum_{i=1}^n \mu[\sigma^i, h^i] \bar{e}_i$  where  $\{\bar{e}_i\}$  is the standard basis for  $R^n$  and  $\mu[\sigma^i, h^i]$  is as in definition (5) above.

**COROLLARY TO THEOREM 1.** *As  $\bar{h}$  tends to zero in  $R_+^n$ ,  $\mu[\bar{\sigma}, \bar{h}]$  converges weakly to  $\bar{\sigma}$ .*

**2. The  $\mathcal{T}$ -variation of a measure.** Given a vector measure  $\bar{\sigma} = (\sigma^1, \dots, \sigma^n)$  in  $\times_{i=1}^n M$  and a sublinear functional  $\mathcal{T}$  on  $R^n$ , Goffman and Serrin [4] obtain a scalar measure  $\mathcal{T}\bar{\sigma}$  in much the same manner as the total variation is defined. We shall state the basic definitions and list three useful theorems from [4] as lemmas.

(6) DEFINITION.  $\mathcal{T}: R^n \rightarrow R$  is a type-A functional on  $R^n$  if it satisfies:

- (a)  $\mathcal{T}(x + y) \leq \mathcal{T}(x) + \mathcal{T}(y)$ .
- (b)  $\mathcal{T}(ax) = a\mathcal{T}(x)$  for positive scalars  $a$ .
- (c) There is  $C > 0$  so that for every  $p$  in  $R^n$  we have  $|\mathcal{T}(p)| \leq C\|p\|$ ;  $\|\cdot\|$  = Euclidean norm.

Note  $\mathcal{T}$  is continuous.

(7) DEFINITION. For a type-A functional  $\mathcal{T}$  on  $R^n$  and  $\bar{\sigma} \in \times_{i=1}^n M$  we define the  $\mathcal{T}$ -variation measure of  $\bar{\sigma}$  to be given by

$$\mathcal{T}\bar{\sigma}(E) = \sup \left\{ \sum_{F \in \pi} \mathcal{T} \circ \bar{\sigma}(F) : \pi \text{ is a finite Borel partition of } E \right\}.$$

LEMMA 1. For  $\bar{\alpha} \perp \bar{\beta}$  and  $\bar{\mu} = \bar{\alpha} + \bar{\beta}$  we have  $\mathcal{T}\bar{\mu} = \mathcal{T}\bar{\alpha} + \mathcal{T}\bar{\beta}$ .

LEMMA 2. Let  $\bar{\sigma}(E) = \int_E a \, dv + \bar{\beta}(E)$  be the Lebesgue decomposition of  $\bar{\sigma}$  with respect to the positive measure  $v$ ; then  $\int_E \mathcal{T} \circ a \, dv + \mathcal{T}\bar{\beta}(E)$  is the Lebesgue decomposition of  $\mathcal{T}\bar{\sigma}$  with respect to  $v$ .

LEMMA 3. If  $\mathcal{T}$  is a positive type-A functional and  $\{\bar{\sigma}_k\}$  converges weakly to  $\bar{\sigma}$ , then for each open set  $G$  we have  $\liminf_k \mathcal{T}\bar{\sigma}_k(G) \geq \mathcal{T}\bar{\sigma}(G)$ .

COROLLARY TO THEOREM 1. As  $h$  tends to zero in  $R^+$ ,  $\mu[\sigma, h]^+$  and  $\mu[\sigma, h]^-$  tend weakly to  $\sigma^+$  and  $\sigma^-$  respectively.

PROOF. Since  $\mu[\sigma, h]$  converges weakly to  $\sigma$ , it suffices to show that  $|\mu[\sigma, h]|$  converges weakly to  $|\sigma|$ . But by Lemma 3 we have for each open  $G$

$$\liminf |\mu[\sigma, h]|(G) \geq |\sigma|(G);$$

hence we need only show  $\limsup |\mu[\sigma, h]|(R^m) \leq |\sigma|(R^m)$ . But,

$$\begin{aligned} |\mu[\sigma, h]|(R^m) &= \int_{R^m} |\kappa[\sigma, h](x)| \, dL(x) \\ &\leq \int_{R^m} \kappa[|\sigma|, h](x) \, dL(x) = \mu[|\sigma|, h](R^m). \end{aligned}$$

Given  $\bar{\sigma} \in \times_{i=1}^n M$ ,  $\bar{h} \in R_+^n$  and  $\mathcal{T}$  as above we denote by  $\mu[\mathcal{T}; \bar{\sigma}, \bar{h}]$  the measure given by  $\int_E \mathcal{T} \circ \kappa[\bar{\sigma}, \bar{h}](x) \, dL(x)$ .

Note that the two "different" measures  $\mu[\mathcal{T}; \bar{\sigma}, \bar{h}]$  and  $\mathcal{T}\mu[\bar{\sigma}, \bar{h}]$  are in fact the same. To see this note that the Lebesgue decomposition of  $\mu[\bar{\sigma}, \bar{h}]$  is given by  $\int_E \kappa[\bar{\sigma}, \bar{h}](x) \, dL(x) + \emptyset(E)$  where  $\emptyset$  = zero measure. Hence by Lemma 2 we have that the decomposition of  $\mathcal{T}\mu[\bar{\sigma}, \bar{h}]$  is given by

$$\int_E \mathfrak{T} \circ \kappa[\bar{\sigma}, \bar{h}](x) dL(x) + \mathfrak{T} \emptyset(E) = \mu[\mathfrak{T}; \bar{\sigma}, \bar{h}](E).$$

In general the measures  $\mathfrak{T}\mu[\bar{\sigma}, \bar{h}]$  do not converge weakly to  $\mathfrak{T}\bar{\sigma}$  in higher dimensions; however, for scalar measures we do have weak convergence. More precisely we state:

**THEOREM 2.** *Let  $\sigma \in M$  and  $(h_k)$  be a sequence of positive real numbers converging to zero; then for any nonnegative type-A functional  $\mathfrak{T}$  on  $R$  we have  $\mathfrak{T}\mu[\sigma, h_k]$  converges weakly to  $\mathfrak{T}\bar{\sigma}$ .*

**PROOF.** The only such  $\mathfrak{T}$  are characterized as

$$\mathfrak{T}(p) = \begin{cases} p\mathfrak{T}(1) & \text{for } p \geq 0, \\ -p\mathfrak{T}(-1) & \text{for } p < 0. \end{cases}$$

Put  $\sigma_k = \mu[\sigma, h_k]$ . By our Corollary to Theorem 1 we have that  $\mathfrak{T}(\sigma_k)^+ = \mathfrak{T}(1)(\sigma_k)^+$  and  $\mathfrak{T}(-\sigma_k)^- = \mathfrak{T}(-1)(-\sigma_k)^-$  converge weakly to  $\mathfrak{T}(1)\sigma^+ = \mathfrak{T}\sigma^+$  and  $\mathfrak{T}(-1)(-\sigma^-) = \mathfrak{T}(-\sigma^-)$  respectively. But  $\sigma_k^+ \perp \sigma_k^-$  and  $\sigma^+ \perp \sigma^-$ , so by Lemma 1 we have

$$\mathfrak{T}\sigma_k = \mathfrak{T}(\sigma_k^+ - \sigma_k^-) = \mathfrak{T}\sigma_k^+ + \mathfrak{T}(-\sigma_k^-)$$

converges weakly to  $\mathfrak{T}\sigma^+ + \mathfrak{T}(-\sigma^-) = \mathfrak{T}(\sigma^+ - \sigma^-) = \mathfrak{T}\sigma$ .

**3. The lim sup formula.** Our main concern with the measures  $\mathfrak{T}\mu[\bar{\sigma}, \bar{h}]$  is their weak behavior as  $\bar{h}$  tends to zero. To this end we shall develop a formula for the lim sup of  $\mathfrak{T}\mu[\bar{\sigma}, \bar{h}]$  as  $\bar{h}$  tends to zero. This will require a refinement in the class of functionals considered.

**DEFINITION.** Each type-A functional  $\mathfrak{T}$  on  $R^n$  induces a type-A functional  $\mathfrak{T}^i$  on the  $i$ th coordinate axis (i.e., on  $R$ ) via the following:

$$\mathfrak{T}^i(p) = \mathfrak{T}(0, \dots, 0, p, 0, \dots, 0)$$

where  $p$  is in the  $i$ th place.

By the sublinearity of  $\mathfrak{T}$ , for each  $(p^1, \dots, p^n)$  we have

$$\mathfrak{T}(p^1, \dots, p^n) \leq \sum_{i=1}^n \mathfrak{T}^i(p^i).$$

We shall say that  $\mathfrak{T}$  is a *type-A\** functional if in addition for all  $\bar{p} = (p^1, \dots, p^n)$  and  $i = 1, \dots, n$  we have  $\mathfrak{T}(\bar{p}) \geq \mathfrak{T}^i(p^i)$ . Denote by  $\mathcal{Q}^*(R^n)$  the nonnegative, type-A\* functionals on  $R^n$ .

**THEOREM 3.** *Let  $\bar{\sigma} \in \times_{i=1}^n M$ ,  $\bar{\sigma} = (\sigma^1, \dots, \sigma^n)$  and  $\mathfrak{T} \in \mathcal{Q}^*(R^n)$ . For each  $i$ , let  $\beta_i$  be the singular part of the decomposition of  $\sigma_i$  with respect to  $L$ . Then the following formula holds:*

$$\limsup_{\bar{h} \rightarrow \bar{0}} \mathfrak{T}\mu[\bar{\sigma}, \bar{h}](Q) = \int_Q \mathfrak{T} \circ \frac{d\bar{\sigma}}{dL}(x) dL + \sum_{i=1}^n \mathfrak{T}^i \beta_i(Q).$$

PROOF. Let  $(\bar{h}_k)$  be an arbitrary sequence tending to zero in  $R^n$ . By using the summability of  $\mathcal{T} \circ d\bar{\sigma}/dL$ , Eggeroff's theorem and the singularity of  $\mathcal{T}^i\beta_i$  and  $L$ , we can define inductively a sequence of sets  $\{E_j\}$  having the following properties:

- (1)  $L(E_j) < 1/j$ .
- (2)  $\int_{E_j} \mathcal{T} \circ (d\bar{\sigma}/dL) dL < 1/j$ .
- (3)  $E_{j+1} \subset E_j$  for  $j = 1, 2, \dots$ .
- (4) On  $Q - E_j$ ,  $\kappa[\sigma^i, h_k^i]$  converges uniformly to  $d\sigma^i/dL$  for each  $i = 1, \dots, n$ .
- (5) For every Borel set  $S$  and all  $j$  we have  $\mathcal{T}^i\beta_i(S) = \mathcal{T}^i\beta_i(S \cap E_j)$ .

PART I (lim sup  $\leq$  formula). For each  $j$  we have

$$\mathcal{T}\mu[\bar{\sigma}, \bar{h}_k](Q) = \int_{Q-E_j} \mathcal{T} \circ \kappa[\bar{\sigma}, \bar{h}_k](x) dL + \int_{E_j} \mathcal{T} \circ \kappa[\bar{\sigma}, \bar{h}_k](x) dL.$$

Let  $I_1$  and  $I_2$  denote the first and second terms respectively. First consider the integral  $I_2$ . By the  $A^*$  property and definitions of  $\mathcal{T}^i$  we get

$$(*) \quad I_2 \leq \sum_{i=1}^n \int_{E_j} \kappa[\mathcal{T}^i\alpha_i, h_k^i](x) dL + \sum_{i=1}^n \int_{E_j} \kappa[\mathcal{T}^i\beta_i, h_k^i](x) dL$$

where  $\alpha_i \ll L$ ,  $\beta_i \perp L$  is the Lebesgue decomposition of  $\sigma_i$ . Since  $\alpha_i \ll L$ , the expression  $\kappa[\mathcal{T}^i\alpha_i, h_k^i](x)$  is the integral average of the summable function  $d\mathcal{T}^i\alpha_i/dL$ . Thus as  $k \rightarrow \infty$  this converges in  $L^1$  to  $d\mathcal{T}^i\alpha_i/dL$ . Hence

$$\limsup \int_{E_j} \kappa[\mathcal{T}^i\alpha_i, h_k^i] dL = \mathcal{T}^i\alpha_i(E_j).$$

Applying Theorem 2 from (\*) we conclude that

$$(**) \quad \limsup I_2 \leq \sum \mathcal{T}^i\alpha_i(E_j) + \sum \mathcal{T}^i\beta_i(Q).$$

On  $Q - E_j$ ,  $\kappa[\bar{\sigma}, \bar{h}_k]$  converges uniformly to  $d\bar{\sigma}/dL$  and  $\mathcal{T}$  is bounded, so by the bounded convergence theorem we get:

$$(***) \quad \limsup I_1 \leq \int_Q \mathcal{T} \circ \frac{d\bar{\sigma}}{dL}(x) dL.$$

Letting  $j \rightarrow \infty$  and noting that  $\mathcal{T}^i\alpha_i(E_j) \rightarrow 0$  (\*\*) and (\*\*\*) give Part I.

PART II (lim sup  $\geq$  formula). Only the case  $n = 3$  will be shown, the general case being the same.

Let  $\bar{h}_k = (1/k, 1/k, 1/k)$ . Given an integer  $m$  and  $\varepsilon > 0$ , there exist integers  $j_1, j_2$  and  $j_3, k_1, k_2$ , and  $k_3$  so that  $m < j_1 < j_2 < j_3$ ,  $m < k_1 < k_2 < k_3$  and

$$\int_{E_{j_1}-E_{j_2}} \mathfrak{T}^1 \circ \kappa[\sigma_1, 1/k_1] dL > \mathfrak{T}^1\beta_1(Q) - \varepsilon,$$

$$\int_{E_{j_2}-E_{j_3}} \mathfrak{T}^2 \circ \kappa[\sigma_2, 1/k_2] dL > \mathfrak{T}^2\beta_2(Q) - \varepsilon,$$

$$\int_{E_{j_3}} \mathfrak{T}^3 \circ \kappa[\sigma_3, 1/k_3] dL > \mathfrak{T}^3\beta_3(Q) - \varepsilon.$$

To establish this recall that we know that  $\int \mathfrak{T}^i \circ \kappa[\sigma_i, 1/k_i] dL$  converges weakly to  $\mathfrak{T}^i\sigma_i$ . Whence on the open sets  $E_j$ , we have

$$\liminf \int \mathfrak{T}^i \circ \kappa[\sigma_i, 1/k_i] dL \geq \mathfrak{T}^i\sigma_i(E_j) \geq \mathfrak{T}^i\beta_i(E_j) = \mathfrak{T}^i\beta_i(Q).$$

There is an integer  $k_i(E_j)$  so that  $k > k_i$  implies that

$$\int_{E_j} \mathfrak{T}^i \circ \kappa[\sigma_i, 1/k] dL > \mathfrak{T}^i\beta_i(Q) - \varepsilon.$$

Given  $m$  take  $j_1 \geq m + 1$ , choose  $k_1 = k_1(E_{j_1}) + m$ . Then

$$\int_{E_{j_1}} \mathfrak{T}^1 \circ \kappa[\sigma_1, 1/k_1] dL > \mathfrak{T}^1\beta_1(Q) - \varepsilon.$$

But for this fixed  $k_1$ , the integrand is summable so there is an integer  $j_2 > j_1$  for which  $L(E_{j_2})$  is sufficiently small to make

$$\int_{E_{j_1}-E_{j_2}} \mathfrak{T}^1 \circ \kappa[\sigma_1, 1/k_1] dL > \mathfrak{T}^1\beta_1(Q) - \varepsilon.$$

Choose  $k_2 > k_2(E_{j_2}) + m + k_1$ ; then

$$\int_{E_{j_2}} \mathfrak{T}^2 \circ \kappa[\sigma_2, 1/k_2] dL > \mathfrak{T}^2\beta_2(Q) - \varepsilon.$$

There is an integer  $j_3 > j_2$  for which  $L(E_{j_3})$  is sufficiently small to ensure that

$$\int_{E_{j_2}-E_{j_3}} \mathfrak{T}^2 \circ \kappa[\sigma_2, 1/k_2] dL > \mathfrak{T}^2\beta_2(Q) - \varepsilon.$$

Fix  $k_3 > k_3(E_{j_3}) + m + k_2$ . Now put  $\bar{k}_m^* = (1/k_1, 1/k_2, 1/k_3)$ ,

$$\mathfrak{T}\mu[\bar{\sigma}, \bar{k}_m^*](Q) = \int_{Q-E_{j_1}} + \int_{E_{j_1}-E_{j_2}} + \int_{E_{j_2}-E_{j_3}} + \int_{E_{j_3}}.$$

But  $\mathfrak{T}^i(p^i) \leq \mathfrak{T}(p^1, \dots, p^n)$  for  $i = 1, \dots, n$  so replacing  $\mathfrak{T} \circ \kappa[\bar{\sigma}, \bar{k}_m^*]$  by smaller integrands in each integral we obtain:

$$\begin{aligned}
\mathfrak{T}_\mu[\bar{\sigma}, \bar{k}_m^*](Q) &\geq \int_{Q-E_j} \mathfrak{T} \circ \kappa[\bar{\sigma}, \bar{k}_m^*] dL + \int_{E_{j_1}-E_{j_2}} \mathfrak{T}^1 \circ \kappa[\sigma_1, 1/k_1] dL \\
&\quad + \int_{E_{j_2}-E_{j_3}} \mathfrak{T}^2 \circ \kappa[\sigma_2, 1/k_2] dL + \int_{E_{j_3}} \mathfrak{T}^3 \circ \kappa[\sigma_3, 1/k_3] dL \\
&\geq \sum_{i=1}^3 \mathfrak{T}^i \beta_i(Q) + \int_{Q-E_j} \mathfrak{T} \circ \kappa[\bar{\sigma}, \bar{k}_m^*] dL - 3\varepsilon.
\end{aligned}$$

Again on  $Q - E_j$  we have uniform convergence, and we get

$$\liminf_m \mathfrak{T}_\mu[\bar{\sigma}, \bar{k}_m^*](Q) \geq \text{formula} - 3\varepsilon - 1/j_1.$$

The existence of such a sequence  $\{\bar{k}_m^*\}$  establishes Part II.

**4. The necessary and sufficient conditions.** For  $\mathfrak{T} \in \mathcal{Q}^*(R^n)$  we can now establish necessary and sufficient conditions for  $\lim \mathfrak{T}_\mu[\bar{\sigma}, \bar{h}]$  to exist and give  $\mathfrak{T}\bar{\sigma}$ . We first give an easy condition on  $\mathfrak{T}$ ; and then, restricting our attention to the Euclidean norm, obtain a deeper condition on the measure  $\bar{\sigma}$ .

**THEOREM 4.** Let  $\mathfrak{T} \in \mathcal{Q}^*(R^m)$  and  $\bar{\sigma} \in \times_{i=1}^n M$ . Then  $\mathfrak{T}_\mu[\bar{\sigma}, \bar{h}](Q)$  converges to  $\mathfrak{T}\bar{\sigma}(Q)$  if and only if  $\mathfrak{T}\bar{\beta}(Q) = \sum_{i=1}^n \mathfrak{T}^i \beta_i(Q)$ , where  $\bar{\beta} = (\beta_1, \dots, \beta_n)$  is the singular part of  $\bar{\sigma}$ .

**PROOF.** By Theorem 3, Lemma 2, and Lemma 3, we have:

$$\begin{aligned}
\int_Q \mathfrak{T} \circ \frac{d\bar{\sigma}}{dL} dL + \mathfrak{T}\bar{\beta}(Q) &= \mathfrak{T}\bar{\sigma}(Q) \leq \liminf_{\bar{h} \rightarrow \bar{\sigma}} \mathfrak{T}_\mu[\bar{\sigma}, \bar{h}](Q) \\
&\leq \limsup_{\bar{h} \rightarrow \bar{\sigma}} \mathfrak{T}_\mu[\bar{\sigma}, \bar{h}](Q) \\
&= \int_Q \mathfrak{T} \circ \frac{d\bar{\sigma}}{dL} dL + \sum_{i=1}^n \mathfrak{T}^i \beta_i(Q).
\end{aligned}$$

If in Theorem 4 we replace  $\mathfrak{T}$  with the Euclidean  $n$ -norm,  $\|\cdot\|$ , we then have:  $\int_Q \|k[\bar{\sigma}, \bar{h}](x)\| dL$  converges to  $\|\bar{\sigma}\|(Q)$  if and only if  $\|\bar{\beta}\|(Q) = \sum_{i=1}^n |\beta_i|(Q)$ .

This yields the equivalent condition on  $\bar{\beta}$ :

**THEOREM 5.** Let  $\bar{\beta} \in \times_{i=1}^n M$ ; then for every Borel set  $E$ ,  $\|\bar{\beta}\|(E) = \sum |\beta_i|(E)$  if and only if for  $i \neq j$ ,  $\beta_i \perp \beta_j$ .

**REMARK.** Since  $\|\bar{\beta}\|$  and  $\sum |\beta_i|$  are both measures, equality on each  $E$  and on  $Q$  is the same in light of the inequality  $\|\bar{\beta}\| \leq \sum |\beta_i|$ .

**PROOF.** Suppose for every Borel set  $E$ ,  $\|\bar{\beta}\|(E) = \sum |\beta_i|(E)$ .

(Special Case)  $n = 2$ ,  $\beta_i \geq 0$ . Since  $\bar{\beta} \ll \|\bar{\beta}\|$ , there is a Radon-Nikodým derivative  $d\bar{\beta}/d\|\bar{\beta}\| = (d\beta_1/d\|\bar{\beta}\|, d\beta_2/d\|\bar{\beta}\|)$  which has modulus 1 everywhere [3]. Hence  $d\|\bar{\beta}\|/d\|\bar{\beta}\| = \|(d\beta_1/d\|\bar{\beta}\|, d\beta_2/d\|\bar{\beta}\|)\| = 1$ . But



$\|\bar{\beta}\|(E) = \beta_1(E) + \beta_2(E)$  implies that except on a set  $S_0$  of  $\|\bar{\beta}\|$  measure zero,

$$\frac{d\beta_1}{d\|\bar{\beta}\|}(x) + \frac{d\beta_2}{d\|\bar{\beta}\|}(x) = \left\| \left[ \frac{d\beta_1}{d\|\bar{\beta}\|}(x), \frac{d\beta_2}{d\|\bar{\beta}\|}(x) \right] \right\|$$

which can happen if and only if one term or the other is zero.

Let  $A_1 = \{x | d\beta_1/d\|\bar{\beta}\| = 0\} - S_0$ ,  $A_2 = \{x | d\beta_2/d\|\bar{\beta}\| = 0\} \cup S_0$ . Since  $\beta_1(A_1) = \int_{A_1} (d\beta_1/d\|\bar{\beta}\|) d\|\bar{\beta}\| = 0$  and

$$\beta_2(A_2) = \int_{A_2 - S_0} \frac{d\beta_2}{d\|\bar{\beta}\|} d\|\bar{\beta}\| + \int_{S_0} \frac{d\beta_2}{d\|\bar{\beta}\|} d\|\bar{\beta}\| = 0 + 0$$

and  $Q = A_1 \cup A_2$  we conclude that  $\beta_1 \perp \beta_2$ .

For  $\beta_i$  signed we note that singularity of  $\beta_i$  and  $\beta_j$  is equivalent to  $|\beta_i|$  and  $|\beta_j|$  reducing to the nonnegative case. For  $n > 2$ , note that  $\sum |\beta_i|(Q) = \|\bar{\beta}\|(Q) \leq [(\|\beta_1, \beta_2\| + \sum_3 |\beta_i|(Q))]$  reducing to  $n = 2$  case.

That the singularity of the components implies the equality of the measures is an easy consequence of the Jordan decomposition of  $Q$  relative to the  $\beta_i$ 's and the triangle inequality.

**5. The area of a nonparametric surface.** Let  $f$  be an  $L_1$  function on the open unit cube in  $R^n$  of type BVT. Then we associate with  $f$  two vector measures  $\bar{\sigma}$  and  $\bar{\sigma}^*$ , where  $\bar{\sigma}$  is the  $n$ -dimensional derivative measure of  $f$  and  $\bar{\sigma}^*$  is the  $(n+1)$ -dimensional measure formed by adjoining Lebesgue measure  $L$  as the first component [i.e.,  $\bar{\sigma} = (\sigma_1, \dots, \sigma_n)$  implies that  $\bar{\sigma}^* = (L, \sigma_1, \dots, \sigma_n)$ ].

It is known [4] that the generalized surface area of  $f$  over  $Q$  is given by  $\|\bar{\sigma}^*\|(Q)$  and that the partial area is given by  $\|\bar{\sigma}\|(Q)$ . By Lemma 2 we have that  $\|\bar{\sigma}\|(Q) = \int_Q \|\bar{\sigma}(x)\| dL + \|\beta\|(Q)$ ,  $\|\bar{\sigma}^*\|(Q) = \int_Q \|\bar{\sigma}^*(x)\| dL(x) + \|\bar{\beta}^*\|(Q)$  where  $\beta$  and  $\bar{\beta}^*$  are the singular parts of  $\bar{\sigma}$  and  $\bar{\sigma}^*$ , respectively. We are now in a position to prove the analogue to L. C. Young's theorem.

**THEOREM 6.** Let  $f$  be an  $L_1$  function on the open unit cube  $Q$  in  $R^n$  of type BVT and let  $\bar{\sigma} = (\sigma_1, \dots, \sigma_n)$  be its distribution derivative measure. For each  $i$ , let  $\beta_i$  be the singular part of the Lebesgue decomposition of  $\sigma_i$ . Then  $\text{Area}(f) = \lim_{h \rightarrow 0} \|\mu[\bar{\sigma}^*, h]\|(Q)$  iff for  $i \neq j$  we have  $\beta_i \perp \beta_j$ .

**PROOF.** This follows from Theorem 5 and our discussion above.

**REMARK.** Note that  $\text{Partial Area}(f) = \lim_{h \rightarrow 0} \|\mu[\bar{\sigma}, h]\|(Q)$  under the same conditions.

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